Perturbative Topological String

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1 Introduction

We shall first have a quick glance on the meaning of perturbative and the how topological string theory comes from the string theory we already familiar with. The note mainly follows Marino's Les Houches lecture, some other references were also referred, the list of references is at the end of the note.

1.1 perturbative and non-perturbative

In short, the concept of perturbative and non-perturbative comes from power series expansion to approach a physical observable. We already know well about how the process of perturbation theory works in classical physics, namely the solution of some complex mechanical systems, and one thing we have to mention is these models or systems have some coupling constant, which should be small enough to enable the well-defining of the theory.

What we called as a perturbative series is generally a formal series(which is written in a series form but not guaranteed to be a convergent series)

$$\phi(g) = \sum_{n \geq 0} a_n g^n \tag{1.1}$$

we connect this series with an actual physical observable with asymptotic approximation

$$F(g) \sim \phi(g) \tag{1.2}$$

and we should note that F(g) can be written in an explicit formation, at least for some range of values of g, which is known as non-perturbative definition of physical observable.

Marino's note gives an example about the perturbation theory we often use in quantum mechanics, but I think the example below is better to show how non-perturbative and perturbative is connected with each other.

Consider a differential equation

$$x^2 y^{'}(x) = y(x) \tag{1.3}$$

we can solve almost any differential equation in series formation:

$$\sum_n n a_n x^{n+1} = \sum_n a_n x^n \tag{1.4}$$

we there obtained

$$y(x) = \sum_n n! x^{n+1} \tag{1.5}$$

which is a formal series, but we can also solve this equation directly as

$$y(x) = Ce^{-1/x} \tag{1.6}$$

which we can recognized as the non perturbative definition of the series. Actually, the process of obtaining this solution from the series is called resurgence and is applied to obtain some physical observable in topological string theory.

1.2 why topological string

Topological string is a simple version of superstring theory, which is directly constructed in algebraic level by connecting a 2d superconformal theory with 2d gravity, same like the original string theory, topological string theory is also defined on a Riemann surface and connect to the geometry of the target space by the classing of maps. The theory is called topological because the 2d superconformal theory in topological string theory is obtained to have topological invariance, the process of obtaining this topological invariance is called topological twisting.

Topological string is originally proposed by a model that maintains the essential information of string theory but can be easier solved, but later it is discovered that topological string can provide information of enumerative geometry on Calabi-Yau threefolds, and they also have close connections to quantum integrable models and precise realizations of large N dualities.

We will not introduce topological string in a typical way, which needs many complicated details that are not important for us. We shall review some geometry details and directly see the topological A and B model in stringy level.

One important insight of topological string theory is the Gromov–Witten invariants occured in string theory. When compactifying the type IIA theory on a CY threefold, it leads to a 4 dimensional supersymmetric theory whose Lagrangian contains moduli-dependent couplings $F_g(t)$, where t is the Kähler moduli of the CY manifold.When these couplings are expanded in the large radius limit, they are of the form

$$F_g(t) = \sum_{\beta \in H_2(X)} N_{g,\beta} e^{-\beta \cdot t} \tag{1.7}$$

where $N_{g,\beta}$ are the GW invariants. Topological string theory is designed to captures precisely the information contained in these couplings. Actually, there are methods to calculate this F_g besides the mirror symmetry method used in topological string theory, which is mentioned in the Appendix.

2 Perturbative topological strings

2.1 target space

Our target space for following discussion is a CY threefold, it is possible for other choices but it is not commonly chosen as a target space. As we already know , a CY threefold is a complex, Kähler,Ricci-flat manifold of complex dimension three.We will denote this target space as M.

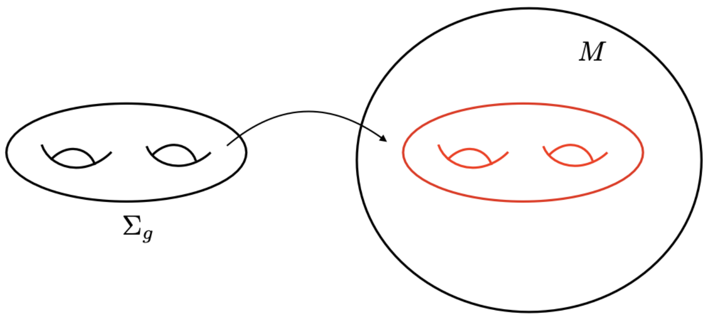

Figure 1. a holomorphic map from a Riemann surface Σ_g into a CY M

It's apparent that a CY threefold may not be compact, but different with when we consider string compactification, we also allow non-compact CY threefold to be our target space here. Actually, we will see that non-compact CY threefolds will be ver important and may be somewhat simpler than compact ones.

The starting point to construct topological string theory is the N = 2 supersymmetric version of the non-linear sigma model, with target space M, as we mentioned above, the different ways of twisting this non-linear sigma model give born of two different models, which is the A and B model. As we connect these models with geometry of target space, A and B model depend to different geometry of the target manifold.

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
=
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} .
$$

Figure 2. The Hodge diamond of CY threefold

As a mathematical property, we shall not prove but use it directly. The topological A model depends on the Kähler parameters of the target space: $h^{1,1}(M) = b_2(M)$, and the topological B model depends on the complex parameters of the target space: $h^{1,2}(M) = \frac{b_3}{2} - 1$. Actually, in mathematical language, we often denote the space described by these parameters of the target space as Kähler (structure) moduli space and complex (structure) moduli space. Of course, we have mirror symmetry, which is a duality between the A model on the CY manifold M and the B model on the mirror CY M', in a mathematical prospective, is a duality between Kähler moduli space and complex moduli space of two CY threefold, or the duality between symplectic geometry and algebraic geometry(complex geometry).

I should mention that those moduli spaces is a direct result of the deformation theory, developed in 1970s. Actually, it's able for us to understand this in a relatively simple way, like P.Candelas do in his famous paper.

Some notations and concepts should be clarified here:

1.We will denote the Kähler parameters and complex parameters as t_i and z_i respectively, sometimes it is denoted as $h^{1,1}(M)$ and $h^{1,2}(M)$ dimensional vector for simplicity.The relation between z and t is called mirror map.

2.What we can observe in topological string theory is the partition function, actually it's the only observable of topological string theory on a CY threefold.

3.But we use free energy, which is the logarithm of the partition function more often. We can calculate free energy as a perturbative series over connected Riemann surfaces.

4.The contribution of a genus g Riemann surfaces to the free energy will be denoted by F_g, and it is a function of the Kähler(respectively, complex) moduli in the A (respectively, B) model.

$$F = \sum_{g=0} F_g g_s^{2g-2} \tag{2.1}$$

where variable g_s is called the topological string coupling constant, is in principle a formal variable, keeping track of the genus of the Riemann surface.However, in string theory this constant has a physical meaning, and measures the strength of the string interaction: when g_s is very small, the contribution to the free energy is dominated by Riemann surfaces of low genus; as g_s becomes large, the contribution of higher genus Riemann surfaces becomes important.

2.2 The A model

To solve topological string theory perturbatively, one has to calculate all F_g, we will perform this process in A model. As we already know in quantum field theories, instantons are the non trivial solutions that have information of the topological information in the theory, so what we consider in topological models are holomorphic maps that represent instantons in the theory:

$$f : \Sigma_g \longrightarrow M \tag{2.2}$$

let $[S_i] \in H_2(M, \mathbb{Z}), i = 1, ..., s$ (we denote $s = b_2(M)$) as a basis for two-homology of M, we can classified these maps topologically by homology class:

$$f_*[(\Sigma_g)] = \sum_{i=1}^{s} d_i[S_i] \in H_2(M, \mathbb{Z}) \tag{2.3}$$

where d_i are integers called the degrees of the map, we often write as a s dimensional vector $\vec{d}$, and the counting of instantons is roughly defined as Gromov-Witten invariants $N_g^{\vec{d}}$(actually the precise definition of GW invariants is complicated, but we can roughly think that GW invariants are related with numbers of such maps(despite that GW invariants is a rational number, rather than a integer)). It may be a fortune for us as it's almost impossible to calculate GW invariants using it's formal definition, for its definition is ver complicated. But we will see GW invariants occur in A model free energy, actually, P.Candelas used mirror symmetry to calculate GW invariants by calculating the B model.

The genus g free energies can be computed as an expansion near the so-called large radius point($t \to \infty$), and they are given by formal power series in e^{-t_i}, also additional contributions as polynomials in t_i.

$$F_0(\vec{t}) = \frac{1}{6} \sum_{i,j,k=1}^{s} a_{i,j,k} t_i t_j t_k + \sum_{\vec{d}} N_{0,\vec{d}} e^{-\vec{d} \cdot \vec{t}} \tag{2.4}$$

$$F_1(\vec{t}) = \sum_{i=1}^{s} b_i t_i + \sum_{\vec{d}} N_{1,\vec{d}} e^{-\vec{d} \cdot \vec{t}} \tag{2.5}$$

$$F_g(\vec{t}) = c_g \chi + \sum_{\vec{d}} N_{g,\vec{d}} e^{-\vec{d} \cdot \vec{t}} \tag{2.6}$$

whicha_{ijk},b_i are obtained by the geometry of M, and c_g is a value depend merely on g.

$$c_g = \frac{(-1)^{g-1} B_{2g} B_{2g-2}}{4g(2g-2)(2g-2)!} \tag{2.7}$$

$$a_{ijk} = -\int_M Im(\mathcal{J})_i \wedge Im(\mathcal{J})_j \wedge Im(\mathcal{J})_k \tag{2.8}$$

$$b_i = \frac{1}{24} \int_M c_2(M) \wedge Im(\mathcal{J})_i \tag{2.9}$$

where $\mathcal{J}$ is the complexified Kähler form, and a_{ijk} is called Yukawa coupling for historical reasons.

Actually we can recognize this form of the free energy as the definition of the free energy, although the original definition is by doing a integral on the moduli space and connecting gravity and topological theories.

The explicit form of free energy is actually a hard question in the history of topological strings, as a brief introduction, I can't provide a detailed explanation, but in short, we can work out the free energy in the following way: as the Kähler moduli is complexified, we have a non-holomorphic free energy that depends both t and $\bar{t}$, and this non-holomorphic free energy has a recursion relation called the holomorphic anomaly equation:

$$\bar{\partial}_{\bar{k}}\mathcal{F}_g = \frac{1}{2}\bar{C}^{ij}_{\bar{k}}(D_iD_j\mathcal{F}_{g-1} + \sum_{r=1}^{g-1} D_iF_rD_j\mathcal{F}_{g-r}) \tag{2.10}$$

which we can solved for some simple models and obtain $\mathcal{F}(t,\bar{t})$, then let $t \to \infty$ to get the holomorphic free energy.

As the total free energy is composed by free energy of different genus

$$F(\vec{t}, g_s) = \sum_{g\geq 0} g_s^{2g-2}F_g(\vec{t}) \tag{2.11}$$

therefore

$$F(\vec{t}, g_s) = F^{(p)}(\vec{t}, g_s) + \sum_{g\geq 0}\sum_{\vec{d}} N_{g,\vec{d}}e^{-\vec{d}\cdot\vec{t}}g_s^{2g-2} \tag{2.12}$$

where

$$F^{(p)}(\vec{t}, g_s) = \frac{1}{6g_s^2}\sum_{i,j,k=1}^{s} a_{ijk}t_it_jt_k + \sum_{i=1}^{s} b_it_i + \chi\sum_{g\geq 2} c_g g_s^{2g-2} \tag{2.13}$$

However, strong evidences are found that imply the total free energy diverges because

$$F_g \sim (2g)! \tag{2.14}$$

In quantum field theory one typically distinguishes between two different sources for the factorial growth of perturbation theory. The first source is due to the growth of Feynman diagrams and is related to instantons. The second source is due to the integration over momenta in some special diagrams, and the corresponding Borel singularities are called "renormalons." In the case of string theory this distinction becomes more subtle. Since there is only one diagram at each genus, we could say that the factorial growth of string theory is due to integration over moduli, and therefore is of the renormalon type.

2.3 The Gopakumar-Vafa representation

When Gopakumar and Vafa noticed that the definition of free energy is a double summation, they realized that it can be resumed to obtain a new invariant which has a geometrical meaning, and this is the Gopakumar-Vafa invariants.

$$F^{GV}(\vec{t}, g_s) = \sum_{g\geq 0} \sum_{\vec{d}} \sum_{w=1}^{\infty} \frac{1}{w} n_g^{\vec{d}} (2sin\frac{wg_s}{2})^{2g-2} e^{-w\vec{d}\cdot\vec{t}} \tag{2.15}$$

It was later found that GV invariants can be interpreted as Euler characteristics of the moduli spaces of D2 branes in the target CY manifold, and because of this, GV invariants are integer numbers. One important property of GV invariants is that for a given degree $\vec{d}$, there is a maximal genus $g_max(\vec{d})$ such $n_g^{\vec{d}} = 0$ for all $g > g_{max}(\vec{d})$.Also, GV invariants sometimes have some recursion relation for simple models. Actually, Gopakumar and Vafa have developed some formula for determining GV invariants directly for simple models like $K3 \times T^2$.

From a direct series expansion, we can see that if one knows the GW invariants, one can determine uniquely the GV invariants. In that sense, the two sets of invariants contain the same information, and thus there exist direct mathematical constructions of the GV invariants as well.

Another fact of the resumation is about the dependence of g_s, for

$$F_{\vec{m}}(g_s) = \sum_{g\geq 0} \sum_{\vec{m}=\vec{d}w} \frac{1}{w} n_g^{\vec{d}} (2sin\frac{wg_s}{2})^{2g-2} \tag{2.16}$$

It is unusual for any rational g_s, there is a minimum degree $\vec{m}_{min}$ such that infinitely many coefficients $F_{\vec{m}}(g_s)$ with $\vec{m} > \vec{m}_{min}$ are singular at that rational value.As a consequence, given any real value of g_s,rational or not, there is a degree starting from which infinitely many coefficients $F_{\vec{m}}(g_s)$ can be made arbitrarily large.

A method named topological vertex was introduced to calculated these $F_m(g_s)$ for simple models quickly, it is not in Marino's note, so I put this content in appendix.

2.4 An example: the resolved conifold

The simplest example of a topological string theory is the one defined on the non-compact CY manifold know as the resolved conifold, which is a plane bundle over the two sphere:

$$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1 \tag{2.17}$$

this may be not a familiar notation for physicist, so we may first have a brief review on conifolds. Actually, a great literature on conifolds is by P.Candelas at 1990, here we also refer to that one.One important fact is a resolved conifold only depends on one parameter, which gives convenience for our computation.

For a conifold

$$\sum_{A=1}^{4} (w^A)^2 = 0 \tag{2.18}$$

the deformation

$$\sum_{A=1}^{4} (w^A)^2 = \epsilon^2 \tag{2.19}$$

results a deformed conifold, and when writing the conifold as $XY - UV = 0$, the resolution

$$\begin{pmatrix} X & U \\ V & Y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0$$

results a resolved conifold.

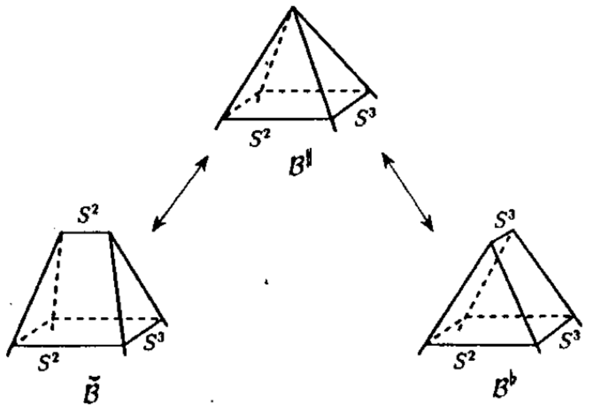

Figure 3. deformed and resolved conifold

in the standard process, we consider the tangent bundle of the resolved conifold and it is composed by tangent and normal bundles of the $\mathbb{P}^1$ manifold.

$$\mathcal{T}(X) = \mathcal{T}_{\mathbb{P}^1} \oplus \mathcal{N}_{\mathbb{P}^1} \tag{2.20}$$

considering that it can be calculated that $c_1(\mathcal{T}_{\mathbb{P}^1}) = 2$ and $c_1(\mathcal{T}_X) = 0$, followed by a theorem by Grothendieck stating that any holomorphic bundle over a $\mathbb{P}_1$ decomposed as a direct sum of line bundles, we have

$$\mathcal{N}_{\mathbb{P}^1} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \tag{2.21}$$

From a complicated calculation in https://arxiv.org/pdf/math/9810173, it is proved that there is only one non-zero GV invariant $n_0^1 = 1$, inserting this into the definition of topological string, we obtain

$$F^{GV}(t; g_s) = \sum_{w=1}^{\infty} \frac{1}{w} \frac{e^{-wt}}{4sin^2(\frac{wg_s}{2})} \tag{2.22}$$

easily we obtain

$$F_0(t) = Li_3(e^{-t}) \tag{2.23}$$

$$F_1(t) = \frac{1}{12} Li_1(e^{-t}) \tag{2.24}$$

$$F_g(t) = \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)!} Li_{3-2g}(e^{-t}), g \geq 2 \tag{2.25}$$

by using the formula

$$Li_{3-2g}(e^{-t}) = \Gamma(2g-2)\sum_{k\in\mathbb{Z}}\frac{1}{(2\pi ki+t)^{2g-2}} \tag{2.26}$$

and the large order behavior of Bernoulli number

$$B_{2g} = \frac{2(-1)^{g+1}(2g)!\zeta(2g)}{(2\pi)^{2g}} \sim \frac{2(-1)^{g+1}(2g)!}{(2\pi)^{2g}} \tag{2.27}$$

we can see that $F_g(t)$ grows doubly-factorially with the genus.

For a special case, a even more simple model is the topological string is near the conifold point defined as $t \to 0$, which we obtained(where $\lambda = it$)

$$F_0(\lambda) = \frac{\lambda^2}{2}(ln(\lambda) - \frac{3}{2}) + ... \tag{2.28}$$

$$F_1(\lambda) = -\frac{1}{12}ln(\lambda) + ... \tag{2.29}$$

$$F_g(\lambda) = \frac{B_{2g}}{2g(2g-2)}\lambda^{2-2g} + ... \tag{2.30}$$

Conifold point is the point where $\mathbb{P}^1$ shrinks to zero size and the free energy is singular, actually these points arise generically in the moduli space of CY manifolds, and they will play an important role in what follows.

2.5 The B model

The problem of calculating the free energies in the B model is very different, since the twisted sigma model localizes to constant maps, so the calculation is in a sense "classical". In the case of genus zero, the problem is completely solved by calculating the periods of the holomorphic 3-form Ω on the mirror CY manifold. Actually, this classicalness enables us to use mirror symmetry to get some thing we expect.

To do the process in genus zero level, one choose a symplectic basis of three-cycles,

$$A^I, B_I \quad I = 0, 1, ..., (h^{2,1})^* \tag{2.31}$$

also these cycles have to satisfy a orthogonal relation

$$\langle A^I, A^J \rangle = \langle B^I, B^J \rangle = 0 \tag{2.32}$$

$$\langle A^I, B_J \rangle = -\langle B_I, A^J \rangle = \delta^I_J \tag{2.33}$$

Integration of Ω over these cycles gives the A and B peroids

$$X^I = \int_{A^I}\Omega, \quad \mathcal{F}_I = \int_{B_I}\Omega \tag{2.34}$$

these periods can define the projective prepotential in a rather indirect way

$$\mathcal{F}_I = \frac{\partial \mathfrak{F}_0}{\partial X^I} \tag{2.35}$$

and we also define the coordinate on the complex moduli space, which is some times called the homogeneous coordinate or flat coordinate

$$t_a = \frac{X^a}{X^0}, \quad a = 1, ..., (h^{2,1})^* \tag{2.36}$$

and thus we get the genus zero free energy from the projective prepotential and the flat coordinates, which is sometimes called the prepotential

$$\mathfrak{F}_0(X^I) = (X^0)^2 F_0(\vec{t}) \tag{2.37}$$

it is not quite a surprise that the prepotential is a global function on the complex moduli space. How is this connected to A model? Actually, from some appropriate choice of the basis of three-cycles, and considering

$$(h^{2,1})* = s \tag{2.38}$$

we can regard the flat coordinates obtained in this way as the K*ä*hle parameters of the A model, thus the easily obtained prepotential is equivalent to the A model genus zero free energy which is defined with GW invariants which we are curious about. This is the classical setting of mirror symmetry practice which is finished on quintic and some simple but non-trivial models.

One important case of CY manifolds is toric CY manifolds and the mirror symmetry on these CY manifolds are usually called local mirror symmetry, their mirror may be written in a form as

$$uv = P(e^x, e^y) \tag{2.39}$$

and $P(e^x, e^y)$ is a polynomial in the exponentiated variables x, y. Also, this mirror map can be viewed as a Riemann surface described by the polynomial

$$P(e^x, e^y) = 0 \tag{2.40}$$

It is proved that this Riemann surface actually inherited some properties of the toric CY threefold, like the nowher vanishing 3form Ω is expressed as a differential on the curve

$$\lambda = y(x)dx \tag{2.41}$$

to follow the same procedure, we can also find a basis of one cycle on the curve

$$\mathcal{A}_a, \mathcal{B}_a, a = 1, ..., g_\Sigma \tag{2.42}$$

where g_Σ represents the genus of the curve, it is natural that independent one cycle depends on the number of genus of the curve.And we can get the flat coordinates directly, because one can always set $X^0 = 1$ at this case

$$t_a = \oint_{\mathcal{A}_a} \lambda, \quad \frac{\partial F_0}{\partial t_a} = \oint_{\mathcal{B}_a} \lambda, \quad a = 1, ..., g_\Sigma \tag{2.43}$$

we need to emphasize that in general $s \geq g_\Sigma$ is true, so additional $s - g_\Sigma$ parameters have to be obtained by considering in additional residues of poles at infinity of the curve, these parameters are called mass parameters.

Another important consequence of using the B model is that there is in fact an infinite family of flat coordinates and genus zero free energies, depending on the choice of a basis of three cycles. This maybe a fact that is quite easy to find, but the thing matters it that different choices can let different flat coordinates be related by symplectic transformations. For a example of local case and $g_\Sigma = 1$,

$$\begin{pmatrix} \partial_{\widetilde{t}}\widetilde{F}_0 \\ \widetilde{t} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \partial_t F_0 \\ t \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\alpha\delta - \beta\gamma = 1 \tag{2.44}$$

we can immediately get

$$\widetilde{F}_0(\widetilde{t}) = F_0(t) - S(t, \widetilde{t}) \tag{2.45}$$

and $S(t, \widetilde{t})$ can be written in a formal form of a order 2 polynomial without constants.But as we have constrains of $S(t, \widetilde{t})$

$$\frac{\partial F_0}{\partial t} = \frac{\partial S}{\partial t} \quad \frac{\partial \widetilde{F}_0}{\partial \widetilde{t}} = -\frac{\partial S}{\partial \widetilde{t}} \tag{2.46}$$

so

$$S(t, \widetilde{t}) = -\frac{\delta}{2\gamma}t^2 + \frac{1}{\gamma}t\widetilde{t} - \frac{\alpha}{2\gamma}\widetilde{t}^2 - \frac{\alpha}{\gamma}t + (\frac{\alpha}{\gamma}a - b)\widetilde{t} \tag{2.47}$$

we called the different choice of three cycles as frames, and these frames are always related by this type of transformations, so they contain the same information, we can choose the most convenient frame to simplifies our calculation.One important frame is the conifold frame which simplified our moduli space to one parameter that is at the surrounding of the conifold point, this frame is rather general because many moduli space of CY manifold is conifold.

For the local case, there is a theorem that allows us to get higher genus free energies, which is previously called the BKMP conjecture

$$exp(\widetilde{F}(\widetilde{t}; g_s)) = \int exp(F(t; g_s) - \frac{1}{g_s^2}S(t, \widetilde{t}))dt \tag{2.48}$$

2.6 A more complicated example: local $\mathbb{P}^2$

We need to look at a rich example to have a better understanding of the B model approach to topological string theory. The mirror curve of $\mathbb{P}^2$ model is given by

$$e^x + e^y + e^{-x-y} + \kappa = 0 \tag{2.49}$$

we define

$$z = \frac{1}{\kappa^3} \tag{2.50}$$

and use the transformation of

$$e^x = -\frac{\kappa}{2} + \frac{bY - a/2}{X + c} \tag{2.51}$$

$$e^y = \frac{a}{X + c} \tag{2.52}$$

we can get a elliptic curve in Weierstrass form, using the definition of the discriminant of elliptic curve, we obtained

$$\Delta(\kappa) = \frac{1 + 27\kappa^3}{\kappa} \tag{2.53}$$

therefore we get the location of the three special points in the curve, we change the discriminant substantially, namely, we have a large radius point $\kappa \to \infty$, an orbifold point $\kappa = 0$, and the conifold point $\kappa = -\frac{1}{3}$, where the discriminant vanishes.

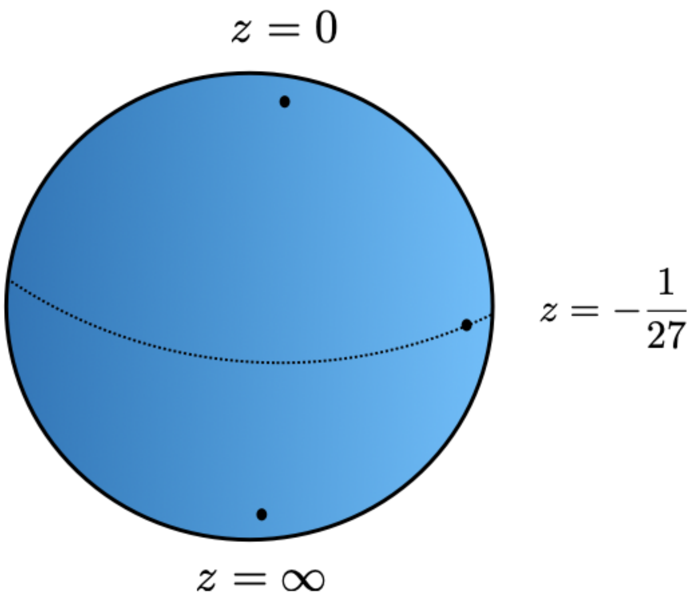

Figure 4. the moduli space of local $\mathbb{P}^2$ with three special points

the detailed discussion of these moduli spaces can be found in Paul.Aspinwall's lecture in 1994.Back to the B model itself, the most convenient way of calculating B model periods which is early proved in the late 80s is to calculate the Picard-Fuchs equation ,which is

$$(\theta^3 - 3z(3\theta + 2)(3\theta + 1)\theta)\Pi = 0 \tag{2.54}$$

and

$$\theta = z\frac{d}{dz} \tag{2.55}$$

for the local $\mathbb{P}^2$ model. The equation has three solutions, and it is obvious that one of them is trivial one, which we will neglect, the other two is denoted as $\varpi_1(z)$ and $\varpi_2(z)$. Which can be written as a complicated composition of generalized hypergeometric function and logarithm. Thus we have

$$t = -\varpi_1(z) \quad \partial_t F_0(t) = \frac{\varpi_2(z)}{6} \tag{2.56}$$

the minus sign here is just a convention. From this we can get that

$$F_0(t) = \frac{t^3}{18} + 3e^- t - \frac{45}{8}e^{-2t} + \frac{244}{9}e^{-3t} - \frac{12333}{64}e^{-4t} + ... \tag{2.57}$$

we should find that for the frame chosen above, when $z \to 0$,we have $e^{-t} \to 0$ which is the large radius limit.So we can also try to choose a conifold frame to get another representation. The period is

$$\lambda(z) = \frac{1}{4\pi}(ln^2(-z) + 2lnz\widetilde{\varpi_1}(z) + \widetilde{\varpi_2}(z)) \tag{2.58}$$

where $\widetilde{\varpi_1}(z)$ and $\widetilde{\varpi_2}(z)$ are generalized hypergeometric functions.The conifold frame is defined as

$$\frac{\partial F_0^c}{\partial \lambda} = -\frac{2\pi}{3}t \pm \frac{2\pi^2 i}{3} \tag{2.59}$$

$$\lambda = \frac{3}{2\pi}\partial_t F_0 \pm \frac{i}{2\pi}t - \frac{\pi}{2} \tag{2.60}$$

where we can obtain

$$F_0^c(\lambda) = \frac{1}{2}\lambda^2(ln(\frac{\lambda}{3^{5/2}}) - \frac{3}{2}) - \frac{\lambda^3}{36\sqrt{3}} + \frac{\lambda^4}{7776} + ... \tag{2.61}$$

which satisfies the general behavior of genus zero free energy, also higher genus can be calculated in other ways and is also verified to be satisfies the general behavior of higher genus free energy.

3 Appendix

3.1 topological strings on open surfaces

The theory of topological strings can be (at least formally) extended to the open case.The natural starting point is to consider maps from a Riemann surface $\Sigma_{g,h}$ of genus g with h holes. Actually, we can find the discussion of this model in one of Witten's paper at 1995. When we consider

$$f : \Sigma_{g,h} \longrightarrow X \tag{3.1}$$

It turns out that the relevant boundary conditions are Dirichlet and given by Lagrangian submanifolds of the Calabi–Yau X, which is a mathematical result.Recall that a Lagrangian submanifold $\mathcal{L}$ is a cycle on which the Kähle form vanishes:

$$J \mid_{\mathcal{L}} = 0 \tag{3.2}$$

If we denote the boundaries of $\Sigma_{g,h}$ as $C_i, \quad i = 1, ..., h$, and pick a Lagrangian submanifold$\mathcal{L}$, we should consider holomorphic maps such that

$$f(C_i) \subset \mathcal{L} \tag{3.3}$$

Once boundary conditions have been specified, we look at holomorphic maps from open Riemann surfaces of genus g and with h holes to the Calabi Yau X, with Dirichlet boundary conditions specified by $\mathcal{L}$. These holomorphic maps are called open string instantons, and can also be classified topologically.

The topological sector of an open string instanton is given by two different kinds of data: the boundary part and the bulk part. For the bulk part, the topological sector is labelled by relative homology classes, since we are requir- ing the boundaries of $f_*[\Sigma_{g,h}]$ to end on $\mathcal{L}$. We set

$$f_*[\Sigma_{g,h}] = \beta \in H_2(X, \mathcal{L}) \tag{3.4}$$

To specify the topological sector of the boundary, we will assume that $b_1(\mathcal{L}) = 1$ so that the first order homology of the Lagrangian submanifold is generated by a non-trivial one cycle which we will denote as γ,we then have

$$f_*(C_i) = w_i \gamma_i, \quad w_i \in \mathbb{Z}, \quad i = 1, .., h \tag{3.5}$$

we can think w_i as winding number associated to the map f restricted to C_i,then we define the open string free energy as

$$F_{w,g}(t) = \sum_{\beta} F_{w,g,\beta} e^{-\beta \cdot t} \tag{3.6}$$

and $F_{w,g,\beta}$ is called open Gromov Witten invariants. The total free energy of open topological string theory is

$$F(V) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \sum_{w_1,\dots,w_h} \frac{i^h}{h!} g_s^{2g-2+h} F_{w,g}(t) Tr V^{w_1} \dots Tr V^{w_h} \tag{3.7}$$

where V is a Hermitian $M \times M$ matrix, the factor of i^h is introduced for convenience and $h!$ is a symmetry factor which takes into account that the holes are indistinguishable. If winding numbers w_i are all positive, the upper formula can be formally written in a simpler form

$$F(V) = \sum_{R} F_R(g_s; t) Tr_R V \tag{3.8}$$

where R is representation.We have also assumed that the boundary conditions are specified by a single Lagrangian submanifold with a single non trivial one cycle. When there are more one-cycles in the geometry, say N, providing possible boundary conditions for the open strings, the above formalism has to be generalized in an obvious way: one introduce different matrix $V_\alpha, \alpha = 1, ..., N$, then the total partition function has the formal form of

$$Z(V_i) = \sum_{R_1,\dots,R_N} Z_{R_1 \dots R_N}(g_s; t) \prod_{\alpha_1}^{N} Tr_{R_\alpha} V_\alpha \tag{3.9}$$

One fact of open string amplitudes is the framing ambiguity which is discovered by Aganagic et al. in 2002. When a vector $f = (p, q)$ is attached to the edge where the submanifold is located and define the symplectic product

$$v \wedge w = v_1 w_2 - v_2 w_1 \tag{3.10}$$

If the original La- grangian submanifold is located at an edge v the choice of framing has to satisfy

$$f \wedge v = 1 \tag{3.11}$$

it is clear that $f - nv$ satisfies the condition for any integer n, and the coefficients Z_R appeared in the total partition function changes as

$$Z_R \rightarrow (-1)^{nl(R)} q^{n\kappa_R/2} Z_R \tag{3.12}$$

where $l(R)$ is defined as the total number of boxes in the young tableaux of the representation, and

$$\kappa_R = l(R) + \sum_i (l_i^2 - 2il_i) \tag{3.13}$$

certainly this can be generalized to

$$Z_{R_1 \ldots R_N} \rightarrow (-1)^{\sum_{\alpha=1}^N n_\alpha l(R_\alpha)} q^{\sum_{\alpha=1}^N n_\alpha \kappa R_\alpha /2} Z_{R_1 \ldots R_N} \tag{3.14}$$

3.2 a very short introduction of topological vertex

As a result in toric geometry, we construct one Lagrangian submanifold in each of the vertices of the toric diagram of $\mathbf{C}^3$,since each of these submanifolds has the topology of $\mathbf{C} \times \mathbf{S}^1$.The total open string partition function in this model will be given by

$$Z(V_i) = \sum_{R_1,R_2,R_3} C_{R_1,R_2,R_3} \prod_{i=1}^{3} Tr_{R_i} V_i \tag{3.15}$$

and the amplitude C_{R_1,R_2,R_3} which is a function of the string coupling constant is called the topological vertex. We then introduce some notion to give the explicit expression for the topological vertex. It is a widely known result that Chern-Simons theory on S^3 is dual to topological string on simple models, such like resolved conifolds. It turns out that the open topological string amplitude for the three Lagrangian submanifolds in $\mathbf{C}^3$ can be written by using only the Chern– Simons invariant of the Hopf link, which we denote as $\mathcal{W}_{R_1 R_2}$ but we won't give a precise definition here considering its complexity. And the limit

$$W_{R_1 R_2} = \lim_{t \to \infty} e^{-\frac{l(R_1)+l(R_2)}{2}} \mathcal{W}_{R_1 R_2} \tag{3.16}$$

is used to obtain topological vertexes.

$$C_{R_1 R_2 R_3} = q^{\frac{\kappa_{R_2}+\kappa_{R_3}}{2}} \sum_{Q_1,Q_3,Q} N_{QQ_1}^{R_1} N_{QQ_3}^{R_3^t} \frac{W_{R_2^t Q_1} W_{R_2 Q_3}}{W_{R_2 0}} \tag{3.17}$$

where $N_{R_1 R_2}^R$ is the Littlewood–Richardson coefficient which gives the multiplicity of R in the tensor product $R_1 \otimes R_2$. Another form of get the explicit form is by introducing Schur polynomials.

Again, we look at the resolved conifold as an example of topological vertex, when using topological vertex, we get

$$Z_{\mathbb{P}^1} = \sum_R C_{00R^t} (-1)^{l(R)} e^{-l(R)t} C_{R00} \tag{3.18}$$

where C_{R00} is a Schur polynomial, explicitly

$$Z_{\mathbb{P}^1} = exp(-\sum_{d=1}^{\infty} \frac{e^{-dt}}{d(q^{d/2} - q^{-d/2})^2}) \tag{3.19}$$

which can be shown to be equivalent to

$$F(g_s; t) = \sum_{d=1}^{\infty} \frac{1}{d(2sin\frac{dg_2}{2})^2} Q^d \tag{3.20}$$

which we are familiar with.

3.3 formal definition and a very short introduction on GW invariants

In order to define Gromov Witten invariants, the starting point is the moduli space of possible metrics (or equivalently, complex structures) on a Riemann surface with punctures, which is the famous Deligne-Mumford space $\bar{M}_{g,n}$ of n-pointed stable curves.Let X be a $K\ddot{a}$hler manifold.The relevant moduli space in Gromov–Witten theory is denoted by

$$\bar{M}_{g,n}(X, \beta) \tag{3.21}$$

where β is a two cycle.Very roughly, a point in $\bar{M}_{g,n}(X, \beta)$ can be written as

$$(f, \Sigma, p_1, ..., p_n) \tag{3.22}$$

which is a combination of a point in a Riemann surface with n punctures together with a choice of complex structure on Σ_g and a holomorphic map with respect to this choice of complex structure and such that $f_*[\Sigma_g] = \beta$. The dimension of this space is

$$(1 - g)(d - 3) + n + \int_{\Sigma_g} f^*(c_1(X)) \tag{3.23}$$

and we also have two maps

$$\pi_1 : \bar{M}_{g,n}(X, \beta) \to X^n : (f, \Sigma, p_1, ..., p_n) \to (f(p_1), ..., f(p_n)) \tag{3.24}$$

$$\pi_2 : \bar{M}_{g,n}(X, \beta) \to \bar{M}_{g,n} : (f, \Sigma, p_1, ..., p_n) \to (\Sigma, p_1, ..., p_n) \tag{3.25}$$

for cohomology classes $\phi_1, ..., \phi_n$ in $H^*(X)$. If we pull back their tensor product to $H^*(\bar{M}_{g,n}(X, \beta))$ by π_1, we get a differential form on the moduli space of maps that we can integrate

$$I_{g,n,\beta}(\phi_1, ..., \phi_n) = \int_{\bar{M}_{g,n}(X,\beta)} \pi_1^*(\phi_1 \otimes ... \otimes \phi_n) \tag{3.26}$$

which is the GW invariants. GW invariants vanishes unless the degree of the form equals the dimension of the moduli space. Therefore, we have the following constraint:

$$\frac{1}{2}\sum_{i=1}^{n} deg(\phi_i) = (1 - g)(d - 3) + n + \int_{\Sigma_g} f^*(c_1(X)) \tag{3.27}$$

for CY threefolds, the condition is always satisfied if ϕ_i have degree 2.

Restricting ourselves to Calabi–Yau threefolds, we have the following mathematical approaches to the computation of Gromov–Witten invariants:

1.*Localization.* This was first proposed by Kontsevich, and requires torus actions in the Calabi–Yau in order to work. Localization provides a priori a complete solution of the theory on toric (hence non-compact) Calabi– Yau manifolds, and reduces the computation of Gromov–Witten invariants to the calculation of Hodge integrals in Deligne–Mumford moduli space. Localization techniques make also possible to solve the theory at genus zero on a wide class of compact manifolds, see for example Cox and Katz (1999) for a review.

2.*Deformation and topological approach.* This has been developed more re- cently and relies on relative Gromov–Witten invariants. It provides a cut-and-paste approach to the calculation of the invariants and seems to be the most powerful approach to higher genus Gromov–Witten invariants in the compact case.

3.$D-brane\quad moduli\quad spaces$. Gromov–Witten invariants can be reformulated in terms of the so-called Gopakumar–Vafa invariants (see Hori et al. (2003) for a summary of these). Heuristic techniques to compute them in terms of Euler characteristics of moduli space of embedded surfaces, and one can recover to a large extent the original information of Gromov–Witten theory. The equivalence between these two invariants remains however conjectural, and a general, rigorous definition of the Gopakumar–Vafa invariants in terms of appropriate moduli spaces is still not known. There is another set of invariants, the so-called Donaldson–Thomas invariants, that are also related to D-brane moduli spaces, which can be rigorously defined and have been conjectured to be equivalent to Gromov–Witten invariants by Maulik, Nekrasov, Okounkov and Pandharipande (2003).

3.4 other method of obtaining $F_g(t)$

One classical method is using large N dualities, large N dualities lead to a computation of the Fg(t) couplings in terms of correlation functions and partition functions in Chern–Simons theory. Although this was formulated originally only for the resolved conifold, one ends up with a general theory which is the theory of the topological vertex, introduced in Aganagic et al. (2005) it also leads to a complete solution on toric Calabi–Yau manifolds. The theory of the topological vertex is closely related to localization and to Hodge integrals, and it can be formulated in a rigorous mathematical way (see Li et al. 2004).

Another more physical way is by using heterotic duality. When the Calabi–Yau manifold has the structure of a K3 fibration, type IIA theory often has a heterotic dual, and the evaluation of Fg(t) restricted to the K3 fiber can be reduced to a one loop integral in heterotic string theory. This leads to explicit, conjectural formulae for Gromov Witten invariants in terms of modular forms.

4 Reference

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this book has a very good introduction on topological vertex.

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this book is extremely recommend for its simplicity and it explains why GW invariants occurs in string theory